**Problem 1:**

Let

\[
S = \left\{ x \in \mathbb{R}^3 \mid x = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.0 \end{bmatrix} + \begin{bmatrix} 1.2 \\ 1.0 \\ 0.0 \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\},
\]

find the orthogonal complement space of \( S \), \( S^\perp \subseteq \mathbb{R}^3 \), and determine an orthonormal basis and dimension for \( S^\perp \). For \( x = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T \in \mathbb{R}^3 \). And find its direct sum representation (i.e., \( x_1 \) and \( x_2 \)) of \( x = x_1 \oplus x_2 \), such that \( x_1 \in S \), \( x_2 \in S^\perp \).
**Problem 2:**
Let $V = F^3$, and let $F$ be the field of rational polynomials. Determine the representation of $v = \begin{bmatrix} s + 2 \\ \frac{1}{s} \\ -2 \end{bmatrix}$ in $(V, F)$ with respect to the basis $\{v^1, v^2, v^3\}$, where

$v^1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v^2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v^3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.
**Problem 3:**

Show that the determinant of the $m \times m$ matrix

$$
\begin{bmatrix}
    s^k & -1 & 0 & \cdots & 0 & 0 \\
    0 & s^{k-1} & -1 & \cdots & 0 & 0 \\
    0 & 0 & s^{k-2} & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & s^{k_2} & -1 \\
    \beta_m(s) & \beta_{m-1}(s) & \beta_{m-2}(s) & \cdots & \beta_2(s) & \beta_1(s)
\end{bmatrix}
$$

is equal to

$$s^n + \beta_1(s)s^{n-k_1} + \beta_2(s)s^{n-k_1-k_2} + \cdots + \beta_m(s)$$

where $n = k_1 + k_2 + \cdots + k_m$ and $\beta_i(s)$ are arbitrary polynomials. (hint: proof by induction)
**Problem 4:**
Given is the system of first-order ordinary differential equation
\[ \dot{x} = t^2 Ax, \]
where \( A \in \mathbb{R}^{n \times n} \) and \( t \in \mathbb{R} \). Determine the state transition matrix \( \Phi(t, t_0) \).
**Problem 5:**

Consider $x(k+1) = A(k)x(k)$. Define

$$\Phi(k,m) = A(k-1)A(k-2)\cdots A(m), \quad \text{for } k > m$$

$$\Phi(m,m) = I$$

Show that, given the initial state $x(m) = x_0$, the state at iteration $k$ is given by $x(k) = \Phi(k,m)x_0$.

If $A$ is independent of $k$, what is $\Phi(k,m)$?