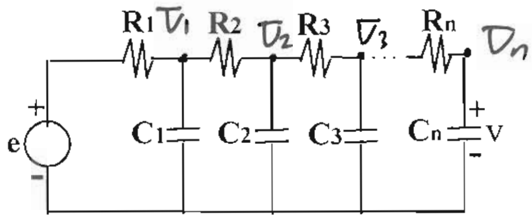
Problem 1: Derive the transfer function $V(s)/E(s)$ for the given RC ladder circuit given below where e is the input source and V is the output response (note $R_1 \neq R_2 \neq \cdots \neq R_n$ and $C_1 \neq C_2 \neq \cdots \neq C_n$)..



1) $\frac{v_1 - e}{R_1} = C_1 \frac{dv_1}{dt} \Rightarrow \frac{\bar{V}_1 - E}{R_1} = C_1 s \bar{V}_1 \Rightarrow \bar{V}_1(1 - C_1R_1S) = E \Rightarrow \frac{\bar{V}_1}{E} = \frac{1}{1 - C_1R_1S}$

2) $\frac{v_2 - v_1}{R_2} = C_2 \frac{dv_2}{dt} \Rightarrow \frac{\bar{V}_2 - \bar{V}_1}{R_2} = C_2 s \bar{V}_2 \Rightarrow \bar{V}_2(1 - C_2R_2S) = \bar{V}_1 \Rightarrow \frac{\bar{V}_2}{\bar{V}_1} = \frac{1}{1 - C_2R_2S}$

3) $\frac{v_3 - v_2}{R_3} = C_3 \frac{dv_3}{dt} \Rightarrow \frac{\bar{V}_3 - \bar{V}_2}{R_3} = C_3 s \bar{V}_3 \Rightarrow \bar{V}_3(1 - C_3R_3S) = \bar{V}_2 \Rightarrow \frac{\bar{V}_3}{\bar{V}_2} = \frac{1}{1 - C_3R_3S}$

⋮

n) $\frac{\bar{V}_n \; \bar{V}_{n-1}}{R_n} = C_n \frac{dv_n}{dt} \Rightarrow \frac{\bar{V}_n - \bar{V}_{n-1}}{R_n} = C_n S \bar{V}_n \Rightarrow \bar{V}_n(1 - C_nR_nS) = \bar{V}_{n-1} \Rightarrow \frac{\bar{V}_n}{\bar{V}_{n-1}} = \frac{1}{1 - C_nR_nS}$

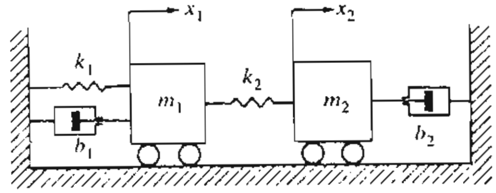
$\bar{V} = \bar{V}_n$

$$\frac{\bar{V}(s)}{E(s)} = \frac{\bar{V}_n(s)}{E(s)} = \frac{\bar{V}_n(s)}{\bar{V}_{n-1}(s)} \cdots \frac{\bar{V}_3(s)}{\bar{V}_2(s)} \cdot \frac{\bar{V}_2(s)}{\bar{V}_1(s)} \cdot \frac{\bar{V}_1(s)}{E(s)}$$

$$= \frac{1}{1 - C_nR_nS} \cdots \frac{1}{1 - C_3R_3S} \cdot \frac{1}{1 - C_2R_2S} \cdot \frac{1}{1 - C_1R_1S}$$

$$= \prod_{i=1}^{n} \frac{1}{1 - C_iR_iS} \quad \#$$

Problem 2: Obtain an *analogous* electrical circuits (using force-current analogy) for the mechanical system shown below.



From Homework #6, Problem #3, we have

$$\begin{cases} m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + b_2\dot{x}_2 - k_2(x_1 - x_2) = 0 \end{cases}$$

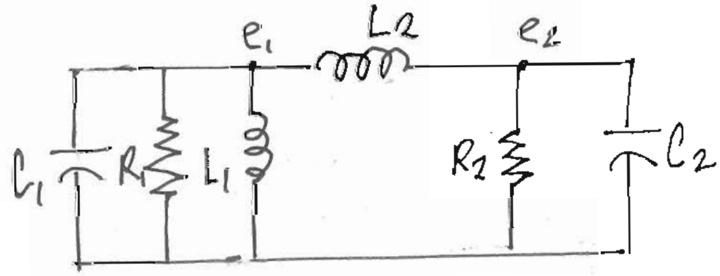
Using the force-current analogy, we have

$$\begin{cases} C_1\ddot{\psi}_1 + \frac{1}{R_1}\dot{\psi}_1 + \frac{1}{L_1}\psi_1 + \frac{1}{L_2}(\psi_1 - \psi_2) = 0 \\ C_2\ddot{\psi}_2 + \frac{1}{R_2}\dot{\psi}_2 - \frac{1}{L_2}(\psi_1 - \psi_2) = 0 \end{cases}$$

Note that $\dot{\psi} = e$, we have

$$\begin{cases} C_1\dot{e}_1 + \frac{1}{R_1}e_1 + \frac{1}{L_1}\int e_1\,dt + \frac{1}{L_2}\int (e_1 - e_2)\,dt = 0 \\ C_2\dot{e}_2 + \frac{1}{R_2}e_2 + \frac{1}{L_2}\int (e_2 - e_1)\,dt = 0 \end{cases}$$

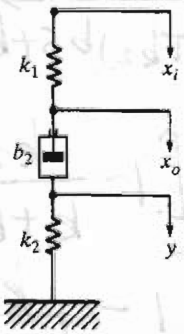
Based on node equations ...



Problem 3:

Derive the transfer function $\frac{X_o(s)}{X_i(s)}$ of the mechanical system shown below. Then obtain he response $x_o(t)$ when the input $x_i(t)$ is a pulse signal given by

$$x_i(t) = \begin{cases} X_i, & 0 < t < t_1 \\ 0, & \text{elsewhere} \end{cases}$$. Assume that $x_o(0-) = 0$.



Equation of Motion of the system:

$$k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$
$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

Rewriting these equations

$$b_2\dot{x}_o + k_1 x_o = k_1\dot{x}_1 + b_2\dot{y}$$
$$b_2\dot{y} + k_2 y = b_2\dot{x}_o$$

Note that $x(0-)=0$, $y(0-)=0$. taking Laplace transform

$$(b_2 s + k_1) X_o(s) = k_1 X_i(s) + b_2 s Y(s)$$
$$(b_2 s + k_2) Y(s) = b_2 s X_o(s)$$

Eliminating $Y(s)$, we have

$$(b_2 s + k_1) X_o(s) = k_1 X_i(s) + b_2 s \frac{b_2 s X_o(s)}{b_2 s + k_2}$$

$$\Rightarrow \frac{X_o(s)}{X_i(s)} = \frac{k_1(b_2 s + k_2)}{(k_1 + k_2) b s + k_1 k_2}$$

Since the input

$$x_i(t) = X_i\left[u(t) - u(\tau_1)\right]$$

$$\Rightarrow X_i(s) = \frac{X_i}{s}\left(1 - e^{-\tau_1 s}\right)$$

$$X_o(s) = \frac{k_1(b_2 s + k_2)}{(k_1 + k_2)bs + k_1 k_2} \cdot \frac{X_i}{s}\left(1 - e^{-\tau_1 s}\right)$$

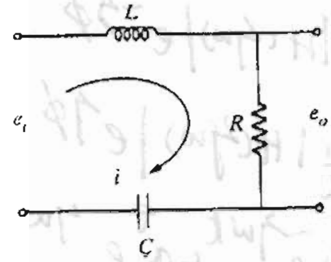
$$x_o(t) = X_i\left[1 - \frac{k_2}{k_1 + k_2} e^{-t/\left(\frac{1}{k_1} + \frac{1}{k_2}\right)b_2}\right]u(t)$$

$$- X_i\left[1 - \frac{k_2}{k_1 + k_2} e^{-(t-\tau_1)/\left(\frac{1}{k_1} + \frac{1}{k_2}\right)b_2}\right] u(t-\tau_1)$$

Problem 4: Consider the electrical circuits shown below. Assume that the input is sinusoidal,

$e_i(t) = E_i \cos \omega t$,

what is the steady state current $i(t)$? Please derive the formula fopr steady state response when the system is subject to an input of $E_i \cos \omega t$.



Assume we have a stable system, $\Rightarrow Re(-S_i) < 0$

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{(s+S_1)(s+S_2)\cdots(s+S_n)}$$

$$Y(s) = H(s)\, X(s) \qquad (\text{where } x(t) = E_i \cos wt)$$

$$= H(s) \frac{E_i s}{s^2 + w^2}$$

$$= \frac{a}{s+jw} + \frac{a^*}{s-jw} + \frac{d_1}{s+S_1} + \frac{d_2}{s+S_2} + \cdots + \frac{d_n}{s+S_n}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$= a e^{-jwt} + a^* e^{jwt} + d_1 e^{-S_1 t} + d_2 e^{-S_2 t} + \cdots + d_n e^{-S_n t}$$

as $t \to \infty$, $\Rightarrow e^{-S_i t}$, $i = 1, \cdots, n \to 0$

Steady state response

$$y(t) = a e^{-jwt} + a^* e^{jwt}$$

$$a = H(s) \frac{E_i s}{s^2 + w^2} (s + jw) \Big|_{s=-jw} = \frac{E_i}{2} H(-jw)$$

$$a^* = H(s) \frac{E_i s}{s^2 + w^2} (s - jw) \Big|_{s=jw} = \frac{E_i}{2} H(jw)$$

Note $H(j\omega) = |H(j\omega)| e^{j\phi}$

$H(-j\omega) = |H(j\omega)| e^{-j\phi}$

$\Rightarrow a = \frac{E_i}{2} |H(j\omega)| e^{-j\phi}$

$a^* = \frac{E_i}{2} |H(j\omega)| e^{j\phi}$

$\Rightarrow y(t) = a e^{-j\omega t} + a^* e^{j\omega t}$

$= |H(j\omega)| E_i \cos(\omega t + \phi)$

$H(s) = \frac{E_o(s)}{E_i(s)} = \frac{R}{Ls + R + \frac{1}{Cs}} = \frac{RCs}{RCs^2 + RCs + 1}$ $\Rightarrow$ $S_{1,2} = \frac{-RC \pm \sqrt{R^2C^2 - 4RC}}{2RC}$

$\Rightarrow$ stable system

$H(j\omega) = \frac{RC(j\omega)}{-RC\omega^2 + jRC\omega + 1}$

$= \frac{jRC\omega}{(1 - RC\omega^2) + j(RC\omega)}$

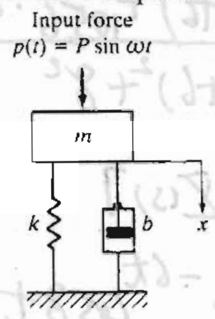
$|H(j\omega)| = \frac{RC\omega}{\sqrt{(1 - RC\omega^2)^2 + (RC\omega)^2}}$

$\angle H(j\omega) = \tan^{-1}\frac{RC\omega}{0} - \tan^{-1}\frac{RC\omega}{1 - RC\omega^2}$

$= \frac{\pi}{2} - \tan^{-1}\frac{RC\omega}{1 - RC\omega^2}$

$\Rightarrow e_o(t) = \frac{RC\omega}{\sqrt{(1 - RC\omega^2)^2 + (RC\omega)^2}} E_i \cos\left(\omega t + \frac{\pi}{2} + \tan^{-1}\frac{RC\omega}{1 - RC\omega^2}\right)$

Problem 5: Consider the mechanical vibratory system shown below. Assume that displacement x is measured from the equilibrium position in the absence of the sinusoidal excitation force. The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$, and the input force $p(t) = P\sin\omega t$ is given at $t = 0$. The numerical values are given as $m = 2$ kg, $b = 24$ N-s/m, $k = 200$ N/m, $P = 5$ N and $\omega = 6$ rad/s. Obtain the complete solution $x(t)$.



Equation of Motion ⇒

$$m\ddot{x} + b\dot{x} + kx = p(t) = P\sin\omega t \quad (x(0)=0,\ \dot{x}(0)=0)$$

Laplace transform ⇒

$$(ms^2+bs+k)X(s) = P(s) = \frac{P\omega}{s^2+\omega^2}$$

Let $m=2,\ b=24,\ k=200,\ P=5,\ \omega=6$

$$(2s^2+24s+200)X(s) = \frac{5\cdot 6}{s^2+6^2}$$

$$X(s) = \frac{15}{(s^2+12s+100)(s^2+6^2)}$$

$$= \frac{as+b}{(s+6)^2+8^2} + \frac{cs+d}{s^2+6^2}$$

$$15 = (as+b)(s^2+6^2) + (cs+d)(s^2+12s+100)$$

$$=(a+c)s^3 + (b+d+12c)s^2 + (36a+12d+100c)s + (36b+100d)$$

$$\Rightarrow a = \frac{3}{29},\ b = \frac{15}{116},\ c = -\frac{9}{464},\ d = \frac{9}{464}$$

$$X(s) = \frac{\frac{9}{464}s + \frac{15}{116}}{(s+6)^2+8^2} + \frac{-\frac{9}{464}s + \frac{3}{29}}{s^2+6^2}$$

$$= \frac{\frac{9}{464}(s+6) + \frac{3}{1856}\cdot 8}{(s+6)^2+8^2} + \frac{-\frac{9}{464}s + \frac{1}{58}\cdot 6}{s^2+6^2}$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$= \frac{9}{464}e^{-6t}\cos 8t + \frac{3}{1856}e^{-6t}\sin 8t$$

$$- \frac{9}{464}\cos 6t + \frac{1}{58}\sin 6t$$