Stability

- homogeneous solution that corresponds to the transient response of the system is governed by the roots of the characteristic equation

$\Rightarrow$

the ~~linear~~ design of linear control systems may be regarded as a problem of arranging the locations of the poles & zeros of the system transfer function

- "Absolute stability" refers to the condition whether the system is stable or unstable. It is a yes or no answer.
- Once the system is found to be stable, it is of interest to determine how stable it is, and the degree of stability is a measure of "relative stability."

Bounded-Input Bounded-Output Stability

With zero initial conditions, the system is said to be BIBO stable, if its output $y(t)$ is bounded to a bounded input $u(t)$

$$y(t) = \int_0^{\infty} h(\tau) u(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_0^{t} h(\tau) u(t-\tau) d\tau \right|$$

$$\leq \int_0^{\infty} |h(\tau)| \, |u(t-\tau)| \, d\tau$$

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if $u(t)$ is bounded, $|u(t)| \le M$
then

$$|y(t)| \le M \int_0^\infty |h(\tau)| d\tau$$

Thus if $y(t)$ is to be bounded,

$$|y(t)| \le N < \infty$$

the following condition must hold

$$M \int_0^\infty |h(\tau)| d\tau \le N < \infty$$

or $$\int_0^\infty |h(\tau)| d\tau \le Q < \infty$$

Relationship between characteristic equation & stability

$$H(s) = \mathcal{L}[h(t)] = \int_0^\infty h(t) e^{-s\tau} d\tau$$

$$|H(s)| = \left| \int_0^\infty h(t) e^{-s\tau} d\tau \right|$$

$$\le \int_0^\infty |h(t)| |e^{-s\tau}| dt$$

Since $|e^{-s\tau}| = |e^{-(\sigma + jw)\tau}| = |e^{-\sigma t}|$

When s assumes a value of a pole of $H(s)$,

$$H(s) = \infty$$

$$\infty \le \int_0^\infty |h(t)| \underline{|e^{-\sigma t}|} dt$$

$$\le \int_0^\infty M |h(t)| dt = \int_0^\infty |h(t)| d\tau$$

which violate the BIBO stability.

⇒ for BIBO stability, the roots of the characteristic equation or the poles of the H(s), cannot be located in the right-half s-plane or on the $j\omega$-axis. In other words, they must all lie in the left-half s-plane.

Stability Condition	Root Values
Asymptotically stable or simply stable	$\sigma_i < 0$ for all i, $i = 1, 2, \ldots, n$. (All the roots are in the left-half s-plane.)
Marginally stable or marginally unstable	$\sigma_i = 0$ for any i for simple roots, and no $\sigma_i > 0$
	For $i = 1, 2, \ldots, n$ (at least one simple root, no multiple-order roots on the $j\omega$-axis, and n roots in the right-half s-plane; note exceptions)
Unstable	$\sigma_i > 0$ for any i, or $\sigma_i = 0$ for any multiple-order root. $i = 1, 2, \ldots, n$ (at least one simple root in the right-half s-plane or at least one multiple-order root on the $j\omega$-axis)

Method to determine stability (without involving root solving)

1. Routh-Hurwitz criterion (algebraic method)
   provide information on the absolute stability test if any of the roots of the characteristic equation lie in the right-half s-plane
2. Nyquist criterion (semigraphical method)
   provide information between the number of poles & zeros of the closed-loop transfer function that are on the right-half s-plane.

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3. Bode diagram (a plot of the magnitude of the loop transfer function $G(j\omega)H(j\omega)$ in dB and the phase of $G(j\omega)H(j\omega)$ in degrees, all versus frequency ω.

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Most linear closed-loop control system :

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

↳ roots ⇔ poles locations

Rouths stability criterion : allows us to determine the number of closed-loop poles that lie in the right-half s-plane without having to factor the polynomial.

1. Write the polynomial in s the following form

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

(assume $a_n \neq 0$)

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient there is a root or roots that are imaginary or that have positive real part. ⇒ system is not stable.

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$$

$$b_3 = \frac{a_1a_6 - a_0a_7}{a_1}$$

$\vdots$ until the remaining ones are all zero

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$$

$$c_3 = \frac{b_1a_7 - a_1b_4}{b_1}$$

$\vdots$

$$d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$$

$$d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$$

$\vdots$

→ the complete array of coefficients is triangular.

→ an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

→ the number of roots with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

⇒ The necessary and sufficient condition that all roots lie in the left-half s-plane is that all the coefficients be 1) positive and 2) all terms in the first column of the array have positive signs.

Example:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

Condition that all roots have negative real parts is given by $a_1 a_2 > a_0 a_3$

Example:

Consider the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{A(s)}{s^4+2s^3+3s^2+4s+5}$$

$\Rightarrow$ $s^4+2s^3+3s^2+4s+5=0$

↳ all coefficients are positive

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

↳ # of changes in sign

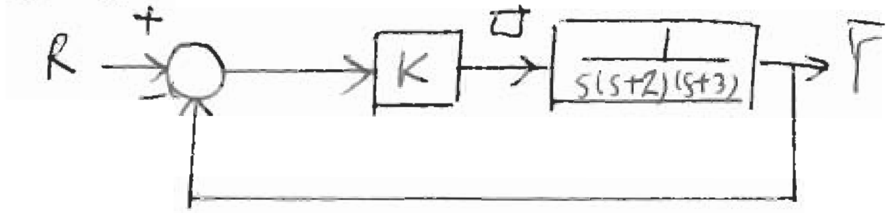
$\Rightarrow$ there are two roots with positive real parts.

Indeed.

roots([1 2 3 4 5])

$\Rightarrow$ 0.2878+1.4161i
0.2878−1.4161i
−1.2878+0.8579i
−1.2878−0.8579i

Example:



$$\frac{Y(s)}{R(s)} = \frac{K}{s(s+2)(s+3)+K} = \frac{K}{s^3+5s^2+6s+K}$$

s^3	1	6
s^2	5	K
s^1	$\frac{30-K}{5}$	
s^0	K	

$K > 0$

$30 - K > 0 \Rightarrow K < 30$

$\Rightarrow \quad 0 < K < 30$

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